1AA3 Review Session 2

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Outline

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Taylor Series

- If we expand a function f at point a, then it must be of the following form

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\ldots$$

  This series is called Taylor series about a. For the special case when $a=0$, we have

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x)^n=f(a)+\frac{f'(a)}{1!}x+\frac{f''(a)}{2!}x^2+\ldots$$

  It has the special name Maclaurin series.

Binomial Series

- If k is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \begin{pmatrix} k \\ n \end{pmatrix} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \ldots$$

Example 1.1

1. Find a power series representation of

$$f(x) = x \tan^{-1} x$$

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+2}$ **(b)** $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ **(c)** $\sum_{n=0}^{\infty} (-1)^n 2n\, x^{2n}$

(d) $\sum_{n=0}^{\infty} \frac{1}{2n+1} x^{2n+2}$ **(e)** $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+2}$

Example 1.2

4. Use the binomial series (or any other method) to find the Maclaurin series for

$$f(x) = \frac{1}{(2+x^2)^2}$$

(a) $\sum_{n=0}^{\infty}(-1)^n\frac{(n+1)}{2^{n+2}}x^{2n}$ **(b)** $\sum_{n=0}^{\infty}(-1)^n\frac{1}{n!\,2^{n+1}}x^{2n}$ **(c)** $\sum_{n=0}^{\infty}(-1)^{n-1}\frac{n}{2^n}x^{2n+1}$

(d) $\sum_{n=0}^{\infty}(-1)^{n-1}\frac{(n+1)}{2^{n+1}}x^{2n+1}$ **(e)** $\sum_{n=0}^{\infty}(-1)^n\frac{n}{2^{n+1}}x^{2n}$

Example 1.3

5. Find the Taylor series for $f(x) = \ln x$ centered at $a = 3$.

(a) $\ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^n}{3^n n}(x-3)^n$

(b) $\ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3^n n}(x-3)^n$

(c) $\ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3^n (n+1)}(x-3)^n$

(d) $\ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3^{n+1} n}(x-3)^n$

(e) $\ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3^n n!}(x-3)^n$

Example 1.4

20. Which of the following is equal to $\binom{1/3}{n}$?

(a) $(-1)^n \dfrac{3\cdot 5\cdot 7\cdot\cdots\cdot(2n+1)}{3^n n!}$ **(b)** $(-1)^{n-1}\dfrac{2\cdot 5\cdot 8\cdot\cdots\cdot(3n-4)}{3^n n!},\ n\geq 2$

(c) $(-1)^n \dfrac{2\cdot 5\cdot 8\cdot\cdots\cdot(3n-1)}{3^n n!}$ **(d)** $(-1)^n\dfrac{2\cdot 5\cdot 8\cdot\cdots\cdot(3n-4)}{n!},\ n\geq 2$

(e) $(-1)^{n-1}\dfrac{3\cdot 5\cdot 7\cdot\cdots\cdot(3n-2)}{3^n n!},\ n\geq 1$

Outline

Separable Equations

- A separable equation can be written in the form

$$\frac{dy}{dx} = g(x)f(y).$$

- To solve this equation we rewrite it as

$$h(y)dy = g(x)dx.$$

- Then we integrate both sides of the equation:

$$\int h(y)dy = \int g(x)dx.$$

- This equation defines y as a function of x. In some cases we may be able to solve for y in terms of x.

Linear Equations

- A first-order linear differential equation can be put into the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where P and Q are continuous functions on given interval.

- The general solution for this equation is given by

$$y(x) = \frac{1}{I(x)}[\int I(x)Q(x)dx + C]$$

where

$$I(x) = \exp\,(\int P(x)dx).$$

Example 2.1

12. Solve the initial value problem

$$(x\ln x)y' + y = x^2 e^x \qquad y(e) = 1$$

(a) $y = \dfrac{xe^x + 1 - e^{e+1}}{\ln x}$ **(b)** $y = \dfrac{e^x(x+1) + 1 - e^e(e+1)}{x}$

(c) $y = \dfrac{xe^x(x-1) + 1 - e^{e+1}(e-1)}{\ln x}$ **(d)** $y = \dfrac{e^x(x-1) + 1 - e^e(e-1)}{\ln x}$

(e) $y = \dfrac{e^x(x^2-1) + 1 - e^e(e^2-1)}{\ln x}$

Example 2.2

14. Find the orthogonal trajectories of the family of curves

$$y = \frac{1}{(x+k)^3}$$

(a) $y = \left(\frac{7}{3}x + C\right)^{9/7}$ **(b)** $y = \left(\frac{9}{7}x + C\right)^{7/3}$ **(c)** $y = \left(\frac{7}{9}x + C\right)^{7/3}$

(d) $y = \left(\frac{9}{7}x + C\right)^{3/7}$ **(e)** $y = \left(\frac{7}{9}x + C\right)^{3/7}$

Example 2.3

16. Solve the initial value problem

$$x^2\frac{dy}{dx} + 2xy = \cos x, \qquad y(\pi) = 0.$$

(a) $y = 3\dfrac{(\cos x) + 1}{x^3}$ **(b)** $y = \dfrac{3\sin x}{x^3}$ **(c)** $y = \dfrac{(\cos x) + 1}{x^2}$
(d) $y = \dfrac{2\sin x}{x^2}$ **(e)** $y = \dfrac{\sin x}{x^2}$

Outline

Parametric Equations

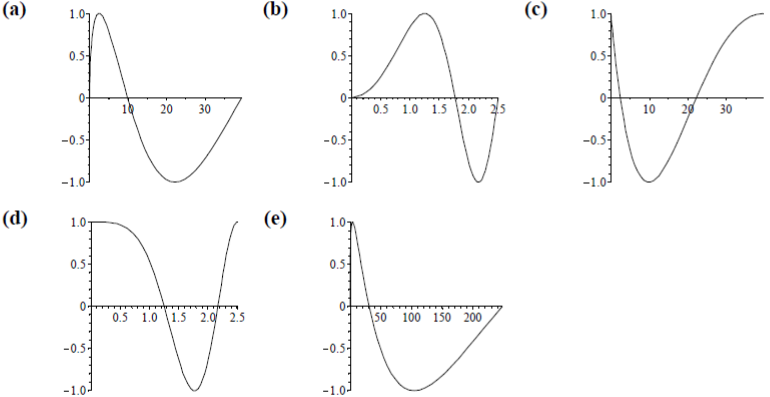
- Suppose that x and y are both given as functions of a third variable t by the equations

$$x = f(t),\ y = g(t).$$

- Each value of t determines a point (x, y), which we can plot in a coordinate plane. In many applications, we can interpret $f(t)$ and $g(t)$ as the position of a particle at time t.

Example 3.1

16. Sketch the parametric curve $x = \sqrt{t}$, $y = \sin t$, $0 \le t \le 2\pi$.



Example 3.2

17. Find the cartesian equation of the parametric curve $x = 1 + e^{2t}$, $y = e^t$.
(a) $x^2 = y^2 - 1$ **(b)** $y^2 = x^2 - 1$ **(c)** $y^2 = \sqrt{x} - 1$ **(d)** $x = y^2 + 1$ **(e)** $y = \sqrt{x} - 1$

Example 3.3

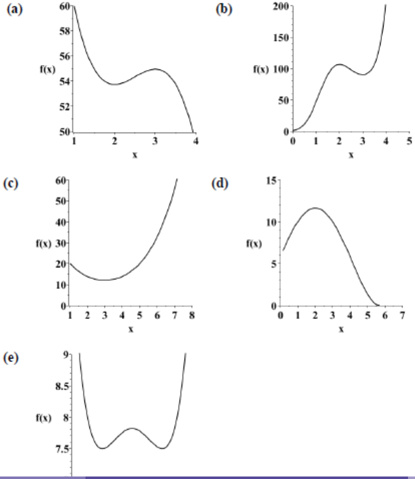
17. Find a cartesian equation of the parametric curve $x = 2\cos t$, $y = 1 + \sin t$.
(a) $\frac{x^2}{4} + (y-1)^2 = 1$ **(b)** $\frac{x}{2} + y = 2$ **(c)** $\frac{x^2}{4} + y^2 = 2$ **(d)** $\frac{x^2}{4} + y^2 = 1$
(e) $\frac{x}{2} + y = 1$

Outline

Example 4.1

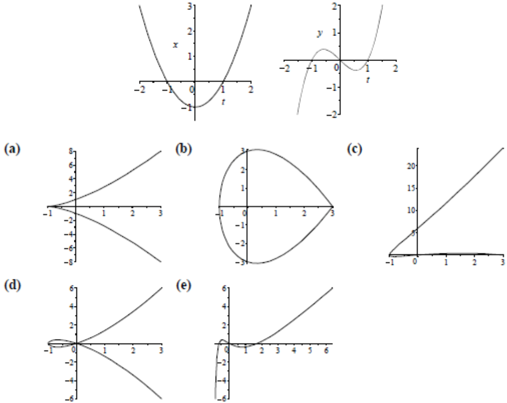
11. Which of the below graphs could be the graph of a function that satisfies the following differential equation?

$$y' = y(x-2)(x-3)$$



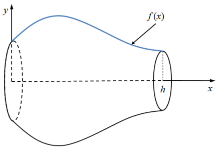
Example 4.2

19. Use the graphs of $x = f(t)$ and $y = g(t)$ below to sketch the parametric curve $x = f(t)$, $y = g(t)$.



Outline

Area of A Surface of Revolution



- The size of the area is

$$S = 2\pi hL$$

- L is length of $f(x)$. By the Arc Length Formula, $L = \int_b^a \sqrt{1+[f'(x)]^2}dx$ which implies $S = \int_b^a 2\pi f(x)\sqrt{1+[f'(x)]^2}dx = \int_b^a 2\pi y\sqrt{1+(\frac{dy}{dx})^2}dx$.

Application to Physics and Engineering

- Hydrostatic Force F is given by

$$F = mg = \rho V g = \rho A d g$$

  where A is surface area and d is depth.

- The pressure P is defined as

$$P = \frac{F}{A} = \rho g d$$

Example 5.1

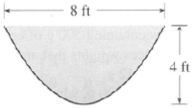
10. Find the area of the surface obtained by rotating the curve $y = \frac{x^3}{6} + \frac{1}{2x}$, $1 \le x \le 2$ about the y-axis.

(a) $\pi(\frac{15}{4} + \ln 2)$ **(b)** $\frac{43}{16}\pi$ **(c)** $\pi(\frac{15}{2} + \ln 2)$ **(d)** $\frac{47}{16}\pi$ **(e)** $\pi(\frac{13}{2} + \ln 3)$

Example 5.2

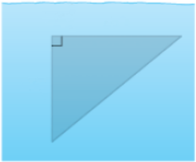
13. A trough is filled with water and its vertical ends have the shape of the parabolic region in the figure to the right. Find the hydrostatic force on one end of the trough.

(a) $\frac{511}{12} \cdot 62.5$ lb **(b)** $\frac{512}{15} \cdot 62.5$ lb **(c)** $\frac{513}{12} \cdot 62.5$ lb
(d) $\frac{512}{13} \cdot 62.5$ lb **(e)** $\frac{511}{15} \cdot 62.5$ lb



Example 5.3

19. A triangular plate with height 5 m and a base of 7 m is submerged vertically in water (whose density is ρ) so that the top is 2 m below the surface. Find the hydrostatic force against one side of the plate.



(a) $\frac{45}{6}\rho g$ **(b)** $\frac{385}{6}\rho g$ **(c)** $\frac{381}{6}\rho g$ **(d)** $\frac{163}{3}\rho g$ **(e)** $50\rho g$

Thank You and Good Luck!